THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

58th Annual American Mathematics Contest 12

AMC 12 CONTEST B

Solutions Pamphlet

Wednesday, FEBRUARY 21, 2007

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

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1. **Answer (E):** The perimeter of each bedroom is $2(12 + 10) = 44$ feet, so the surface to be painted in each bedroom has an area of $44 \cdot 8 - 60 = 292$ square feet. Since there are 3 bedrooms, Isabella must paint $3 \cdot 292 = 876$ square feet.

2. **Answer (B):** The student used $120/30 = 4$ gallons on the trip home and $120/20 = 6$ gallons on the trip back to school. So the average gas mileage for the round trip was

$$\frac{240 \text{ miles}}{10 \text{ gallons}} = 24 \text{ miles per gallon.}$$

3. **Answer (D):** Since $OA = OB = OC$, triangles AOB, BOC, and COA are all isosceles. Hence

$$\angle ABC = \angle ABO + \angle OBC = \frac{180^\circ - 140^\circ}{2} + \frac{180^\circ - 120^\circ}{2} = 50^\circ.$$

OR

Since

$$\angle AOC = 360^\circ - 140^\circ - 120^\circ = 100^\circ,$$

the Central Angle Theorem implies that

$$\angle ABC = \frac{1}{2}\angle AOC = 50^\circ.$$

4. **Answer (B):** Because 3 bananas cost as much as 2 apples, 18 bananas cost as much as 12 apples. Because 6 apples cost as much as 4 oranges, 12 apples cost as much as 8 oranges. Therefore 18 bananas cost as much as 8 oranges.

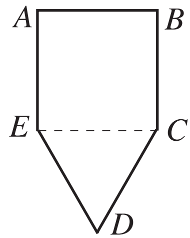
5. **Answer (D):** Sarah will receive 4.5 points for the three questions she leaves unanswered, so she must earn at least $100 - 4.5 = 95.5$ points on the first 22 problems. Because

$$15 < \frac{95.5}{6} < 16,$$

she must solve at least 16 of the first 22 problems correctly. This would give her a score of 100.5.

6. **Answer (D):** The perimeter of the triangle is $5 + 6 + 7 = 18$, so the distance that each bug crawls is 9. Therefore $AB + BD = 9$, and $BD = 4$.

7. **Answer (E):** Because $AB = BC = EA$ and $\angle A = \angle B = 90°$, quadrilateral $ABCE$ is a square, so $\angle AEC = 90°$.



Also $CD = DE = EC$, so $\triangle CDE$ is equilateral and $\angle CED = 60°$. Therefore

$$\angle E = \angle AEC + \angle CED = 90° + 60° = 150°.$$

8. **Answer (D):** Tom's age N years ago was $T - N$. The sum of his three children's ages at that time was $T - 3N$. Therefore $T - N = 2(T - 3N)$, so $5N = T$ and $T/N = 5$. The conditions of the problem can be met, for example, if Tom's age is 30 and the ages of his children are 9, 10, and 11. In that case $T = 30$ and $N = 6$.

9. **Answer (A):** Let $u = 3x - 1$. Then $x = (u+1)/3$, and

$$f(u) = \left(\frac{u+1}{3}\right)^2 + \frac{u+1}{3} + 1 = \frac{u^2+2u+1}{9} + \frac{u+1}{3} + 1 = \frac{u^2+5u+13}{9}.$$

In particular,

$$f(5) = \frac{5^2 + 5 \cdot 5 + 13}{9} = \frac{63}{9} = 7.$$

OR

The value of $3x - 1$ is 5 when $x = 2$. Thus

$$f(5) = f(3 \cdot 2 - 1) = 2^2 + 2 + 1 = 7.$$

10. **Answer (C):** Let g be the number of girls and b the number of boys initially in the group. Then $g = 0.4(g + b)$. After two girls leave and two boys arrive, the size of the entire group is unchanged, so $g - 2 = 0.3(g + b)$. The solution of the system of equations

$$g = 0.4(g + b) \quad \text{and} \quad g - 2 = 0.3(g + b)$$

is $g = 8$ and $b = 12$, so there were initially 8 girls.

OR

After two girls leave and two boys arrive, the size of the group is unchanged. So the two girls who left represent $40\% - 30\% = 10\%$ of the group. Thus the size of the group is 20, and the original number of girls was 40% of 20, or 8.

11. **Answer (D):** Let x be the degree measure of $\angle A$. Then the degree measures of angles B, C, and D are $x/2$, $x/3$, and $x/4$, respectively. The degree measures of the four angles have a sum of 360, so

$$360 = x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{25x}{12}.$$

Thus $x = (12 \cdot 360)/25 = 172.8 \approx 173$.

12. **Answer (C):** Let N be the number of students in the class. Then there are $0.1N$ juniors and $0.9N$ seniors. Let s be the score of each junior. The scores totaled $84N = 83(0.9N) + s(0.1N)$, so

$$s = \frac{84N - 83(0.9N)}{0.1N} = 93.$$

Note: In this problem, we could assume that the class has one junior and nine seniors. Then

$$9 \cdot 83 + s = 10 \cdot 84 = 9 \cdot 84 + 84 \quad \text{and} \quad s = 9(84 - 83) + 84 = 93.$$

13. **Answer (D):** The light completes a cycle every 63 seconds. Leah sees the color change if and only if she begins to look within three seconds before the change from green to yellow, from yellow to red, or from red to green. Thus she sees the color change with probability $(3 + 3 + 3)/63 = 1/7$.

14. **Answer (D):** Let the side length of $\triangle ABC$ be s. Then the areas of $\triangle APB$, $\triangle BPC$, and $\triangle CPA$ are, respectively, $s/2$, s, and $3s/2$. The area of $\triangle ABC$ is the sum of these, which is $3s$. The area of $\triangle ABC$ may also be expressed as $(\sqrt{3}/4)s^2$, so $3s = (\sqrt{3}/4)s^2$. The unique positive solution for s is $4\sqrt{3}$.

15. **Answer (E):** The terms involving odd powers of r form the geometric series $ar + ar^3 + ar^5 + \cdots$. Thus

$$7 = a + ar + ar^2 + \cdots = \frac{a}{1-r},$$

and

$$3 = ar + ar^3 + ar^5 + \cdots = \frac{ar}{1-r^2} = \frac{a}{1-r} \cdot \frac{r}{1+r} = \frac{7r}{1+r}.$$

Therefore $r = 3/4$. It follows that $a/(1/4) = 7$, so $a = 7/4$ and

$$a + r = \frac{7}{4} + \frac{3}{4} = \frac{5}{2}.$$

OR

The sum of the terms involving even powers of r is $7-3=4$. Therefore

$$3 = ar + ar^3 + ar^5 + \cdots = r(a + ar^2 + ar^4 + \cdots) = 4r,$$

so $r = 3/4$. As in the first solution, $a = 7/4$ and $a + r = 5/2$.

16. **Answer (A):** Let r, w, and b be the number of red, white, and blue faces, respectively. Then (r, w, b) is one of 15 possible ordered triples, namely one of the three permutations of $(4,0,0)$, $(2,2,0)$, or $(2,1,1)$, or one of the six permutations of $(3,1,0)$. The number of distinguishable colorings for each of these ordered triples is the same as for any of its permutations. If $(r,w,b) = (4,0,0)$, then exactly one coloring is possible. If $(r,w,b) = (3,1,0)$, the tetrahedron can be placed with the white face down. If $(r,w,b) = (2,2,0)$, the tetrahedron can be placed with one white face down and the other facing forward. If $(r,w,b) = (2,1,1)$, the tetrahedron can be placed with the white face down and the blue face forward. Therefore there is only one coloring for each ordered triple, and the total number of distinguishable colorings is 15.

17. **Answer (D):** Because $b < 10^b$ for all $b > 0$, it follows that $\log_{10} b < b$. If $b \geq 1$, then $0 < (\log_{10} b)/b^2 < 1$, so a cannot be an integer. Therefore $0 < b < 1$, so $\log_{10} b < 0$ and $a = (\log_{10} b)/b^2 < 0$. Thus $a < 0 < b < 1 < 1/b$, and the median of the set is b.

    Note that the conditions of the problem can be met with $b = 0.1$ and $a = -100$.

18. **Answer (C):** Let N^2 be the smaller of the two squares. Then the difference between the two squares is $(N+1)^2 - N^2 = 2N+1$. The given conditions state that

$$100a + 10b + c = N^2 + \frac{2N+1}{3} \quad \text{and} \quad 100a + 10c + b = N^2 + \frac{2(2N+1)}{3}.$$

Subtraction yields $9(c-b) = (2N+1)/3$, from which $27(c-b) = 2N+1$. If $c - b = 0$ or 2, then N is not an integer. If $c - b \geq 3$, then $N \geq 40$, so N^2 is not a three-digit integer. If $c - b = 1$, then $N = 13$. The numbers that are, respectively, one third of the way and two thirds of the way from 13^2 to 14^2 are 178 and 187, so $a + b + c = 1 + 7 + 8 = 16$.

19. **Answer (A):** Let $\theta = \angle ABC$. The base of the cylinder is a circle with circumference 6, so the radius of the base is $6/(2\pi) = 3/\pi$. The height of the cylinder is the altitude of the rhombus, which is $6\sin\theta$. Thus the volume of the cylinder is

$$6 = \pi\left(\frac{3}{\pi}\right)^2 (6\sin\theta) = \frac{54}{\pi}\sin\theta,$$

so $\sin\theta = \pi/9$.

20. **Answer (D):** Two vertices of the first parallelogram are at $(0, c)$ and $(0, d)$. The x-coordinates of the other two vertices satisfy $ax + c = bx + d$ and $ax + d = bx + c$, so the x-coordinates are $\pm(c - d)/(b - a)$. Thus the parallelogram is composed of two triangles, each of which has area

$$9 = \frac{1}{2} \cdot |c - d| \cdot \left|\frac{c - d}{b - a}\right|.$$

It follows that $(c - d)^2 = 18|b - a|$. By a similar argument using the second parallelogram, $(c + d)^2 = 72|b - a|$. Subtracting the first equation from the second yields $4cd = 54|b - a|$, so $2cd = 27|b - a|$. Thus $|b - a|$ is even, and $a + b$ is minimized when $\{a, b\} = \{1, 3\}$. Also, cd is a multiple of 27, and $c + d$ is minimized when $\{c, d\} = \{3, 9\}$. Hence the smallest possible value of $a + b + c + d$ is $1 + 3 + 3 + 9 = 16$. Note that the required conditions are satisfied when $(a, b, c, d) = (1, 3, 3, 9)$.

21. **Answer (A):** Because $3^6 = 729 < 2007 < 2187 = 3^7$, it is convenient to begin by counting the number of base-3 palindromes with at most 7 digits. There are two palindromes of length 1, namely 1 and 2. There are also two palindromes of length 2, namely 11 and 22. For $n \geq 1$, each palindrome of length $2n + 1$ is obtained by inserting one of the digits 0, 1, or 2 immediately after the nth digit in a palindrome of length $2n$. Each palindrome of length $2n+2$ is obtained by similarly inserting one of the strings 00, 11, or 22. Therefore there are 6 palindromes of each of the lengths 3 and 4, 18 of each of the lengths 5 and 6, and 54 of length 7. Because the base-3 representation of 2007 is 2202100, that integer is less than each of the palindromes 2210122, 2211122, 2212122, 2220222, 2221222, and 2222222. Thus the required total is $2+2+6+6+18+18+54-6 = 100$.

22. **Answer (A):** Imagine a third particle that moves in such a way that it is always halfway between the first two. Let D, E, and F denote the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, and let X, Y, and Z denote the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, respectively. When the first particle is at A, the second is at D and the third is at X. When the first particle is at F, the second is at C and the third is at Z. Between those two instants, both coordinates of the first two particles are linear functions of time. Because the average of two linear functions is linear, the third particle traverses $\overline{XZ}$. Similarly, the third particle traverses $\overline{ZY}$ as the first traverses $\overline{FB}$ and the second traverses $\overline{CE}$. Finally, as the first particle traverses $\overline{BD}$ and the second traverses $\overline{EA}$, the third traverses $\overline{YX}$. As the first two particles return to A and D, respectively, the third makes a second circuit of $\triangle XYZ$.

If O is the center of $\triangle ABC$, then by symmetry O is also the center of equilateral $\triangle XYZ$. Note that

$$OZ = OC - ZC = \frac{2}{3}CF - \frac{1}{2}CF = \frac{1}{6}CF,$$

so the ratio of the area of $\triangle XYZ$ to that of $\triangle ABC$ is

$$\left(\frac{OZ}{OC}\right)^2 = \left(\frac{\frac{1}{6}CF}{\frac{2}{3}CF}\right)^2 = \frac{1}{16}.$$

23. **Answer (A):** Let the triangle have leg lengths a and b, with $a \le b$. The given condition implies that

$$\frac{1}{2}ab = 3\left(a + b + \sqrt{a^2 + b^2}\right),$$

so

$$ab - 6a - 6b = 6\sqrt{a^2 + b^2}.$$

Squaring both sides and simplifying yields

$$ab\,(ab - 12a - 12b + 72) = 0,$$

from which

$$(a - 12)(b - 12) = 72.$$

The positive integer solutions of the last equation are $(a, b) = (3, 4)$, $(13, 84)$, $(14, 48)$, $(15, 36)$, $(16, 30)$, $(18, 24)$, and $(20, 21)$. However, the solution $(3, 4)$ is extraneous, and there are six right triangles with the required property.

Query: Why do the given conditions imply that the hypotenuse also has integer length?

24. **Answer (A):** Let $u = a/b$. Then the problem is equivalent to finding all positive rational numbers u such that

$$u + \frac{14}{9u} = k$$

for some integer k. This equation is equivalent to $9u^2 - 9uk + 14 = 0$, whose solutions are

$$u = \frac{9k \pm \sqrt{81k^2 - 504}}{18} = \frac{k}{2} \pm \frac{1}{6}\sqrt{9k^2 - 56}.$$

Hence u is rational if and only if $\sqrt{9k^2 - 56}$ is rational, which is true if and only if $9k^2 - 56$ is a perfect square. Suppose that $9k^2 - 56 = s^2$ for some positive integer s. Then $(3k - s)(3k + s) = 56$. The only factors of 56 are 1, 2, 4, 7, 8, 14, 28, and 56, so $(3k - s, 3k + s)$ is one of the ordered pairs $(1, 56)$, $(2, 28)$, $(4, 14)$, or $(7, 8)$. The cases $(1, 56)$ and $(7, 8)$ yield no integer solutions. The cases $(2, 28)$ and $(4, 14)$ yield $k = 5$ and $k = 3$, respectively. If $k = 5$, then $u = 1/3$ or $u = 14/3$. If $k = 3$, then $u = 2/3$ or $u = 7/3$. Therefore there are four pairs (a, b) that satisfy the given conditions, namely $(1, 3)$, $(2, 3)$, $(7, 3)$, and $(14, 3)$.

OR

Rewrite the equation

$$\frac{a}{b} + \frac{14b}{9a} = k$$

in two different forms. First, multiply both sides by b and subtract a to obtain

$$\frac{14b^2}{9a} = bk - a.$$

Because a, b, and k are integers, $14b^2$ must be a multiple of a, and because a and b have no common factors greater than 1, it follows that 14 is divisible by a. Next, multiply both sides of the original equation by $9a$ and subtract $14b$ to obtain

$$\frac{9a^2}{b} = 9ak - 14b.$$

This shows that $9a^2$ is a multiple of b, so 9 must be divisible by b. Thus if (a, b) is a solution, then $b = 1$, 3, or 9, and $a = 1$, 2, 7, or 14. This gives a total of twelve possible solutions (a, b), each of which can be checked quickly. The only such pairs for which

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer are when (a, b) is $(1, 3)$, $(2, 3)$, $(7, 3)$, or $(14, 3)$.

25. **Answer (C):** Introduce a coordinate system in which $D = (-1, 0, 0)$, $E = (1, 0, 0)$, and $\triangle ABC$ lies in a plane $z = k > 0$. Because $\angle CDE$ and $\angle DEA$ are right angles, A and C are located on circles of radius 2 centered at E and D in the planes $x = 1$ and $x = -1$, respectively. Thus $A = (1, y_1, k)$ and $C = (-1, y_2, k)$, where $y_j = \pm\sqrt{4 - k^2}$ for $j = 1$ and 2. Because $AC = 2\sqrt{2}$, it follows that $(1 - (-1))^2 + (y_1 - y_2)^2 = \left(2\sqrt{2}\right)^2$. If $y_1 = y_2$, there is no solution, so $y_1 = -y_2$. It may be assumed without loss of generality that $y_1 > 0$, in which case $y_1 = 1$ and $y_2 = -1$. It follows that $k = \sqrt{3}$, so $A = (1, 1, \sqrt{3})$, $C = \left(-1, -1, \sqrt{3}\right)$, and B is one of the points $\left(1, -1, \sqrt{3}\right)$ or $\left(-1, 1, \sqrt{3}\right)$. In the first case, $BE = 2$ and $\overline{BE} \perp \overline{DE}$. In the second case, $BD = 2$ and $\overline{BD} \perp \overline{DE}$. In either case, the area of $\triangle BDE$ is $(1/2)(2)(2) = 2$.

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